Notes on the deterministic model in Judd & Guu, 1993. Perturbation solution methods for economic growth models. In Varian, H.R. (Ed.), *Economic and Financial Modeling with Mathematica*, Springer, New York.

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The paper solves a simple optimal growth model with perturbation. This document contains some clarifying notes for the Python code based on this paper. For details see the original paper by Judd and Guu.

The optimization problem is as follows:

$$V\left(k_0\right) = \max_c \int_0^\infty e^{-\rho t} u\left(c\right) \, dt \tag{1a}$$

$$\frac{dk}{dt} = f\left(k\right) - c \tag{1b}$$

$$k\left(0\right) = k_0 \tag{1c}$$

The Bellman Equation of this problem is:

$$\rho V\left(k\right) = u\left(C\left(k\right)\right) + V'\left(k\right)\left(f\left(k\right) - C\left(k\right)\right) \tag{2}$$

First-order condition:

$$V'\left(k\right) = u'\left(C\left(k\right)\right) \tag{3}$$

Differentiate (2) and (3) with respect to k to get

$$u' C_k + V''\left(f - C\right) + V'\left(f_k - C_k\right) - \rho V' = 0 \tag{4a}$$

$$V'' = u'' C_k \tag{4b}$$

Insert (3) and (4b) in (4a) to get

$$0 = C_k(f - C) - \frac{u'}{u''}\left(\rho - f_k\right) \Rightarrow 0 = C_k f - C_k C - \frac{u'}{u''}\rho + \frac{u'}{u''} f_k \tag{5}$$

We assume that $u = \frac{1}{1+\gamma} c^{1+\gamma}$, so that

$$\frac{u'}{u''} = \frac{C^\gamma}{\gamma C^{\gamma-1}} = \frac{C}{\gamma} \tag{6}$$

Inserting this in the Bellman Equation gives

$$0 = C_k f - C_k C - \frac{C}{\gamma}\rho + \frac{C}{\gamma} f_k \tag{7}$$

Multiply by γ and rearrange to get

$$0 = -C\,\rho - \gamma\,C\,C_k + \gamma\,f\,C_k + C\,f_k \tag{8}$$

This is the Bellman Equation in the Mathematica code in the paper and in the Python model. The production function is $f = \frac{\rho}{\alpha}k^{\alpha}$.

The consumption function is approximated by a Taylor polynomial:

$$\hat{C}\left(k\right) = \sum_{i=0}^{n} \frac{1}{i!}\bar{C}_{(ik)}\left(k - \bar{k}\right)^{i} \tag{9}$$

where $\bar{C} \doteq C\left(\bar{k}\right)$ denotes the consumption function in the steady state.

The residual function is constructed by substituting $\hat{C}\left(k\right)$ in the Bellman Equation, and normalizing by $\rho\,\bar{C}$:

$$R\left(k\right) = \frac{-\hat{C}\left(k\right)\,\rho - \gamma\,\hat{C}\left(k\right)\,\hat{C}_k\left(k\right) + \gamma\,f\,\hat{C}_k\left(k\right) + \hat{C}\left(k\right)\,f_k}{\rho\,\bar{C}} \tag{10}$$