Notes on the stochastic model in Judd & Guu, 1993. Perturbation solution methods for economic growth models. In Varian, H.R. (Ed.), *Economic and Financial Modeling with Mathematica*, Springer, New York.

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The paper solves a simple stochastic optimal growth model with perturbation. This document contains some clarifying notes for the Python code based on this paper. For details see the original paper by Judd and Guu.

The optimization problem is as follows:

$$V\left(k_0\right) = \max_c \mathbb{E}\left\{\int_0^\infty e^{-\rho\, t} u\left(c\right)\, dt\right\} \tag{1a}$$

$$dk = \left(f\left(k\right) - c\right)\, dt + \frac{1}{2}\sqrt{\sigma}\, k\, dz \tag{1b}$$

$$k\left(0\right) = k_0 \tag{1c}$$

The Bellman Equation of this problem is:

$$0 = \max_c \left[-\rho\, V\left(k\right) + u\left(c\right) + V'\left(k\right)\left(f\left(k\right) - c\right) + \sigma\, k^2 V''\left(k\right)\right] \tag{2}$$

First-order condition:

$$V'\left(k\right) = u'\left(C\left(k\right)\right) \tag{3}$$

Differentiate (2) and (3) with respect to k:

$$-\rho\, V_k + u'\, C_k + V_{kk}\left(f - C\right) + V_k\, \left(f_k - C_k\right) + 2\,\sigma\, k\, V_{kk} + \sigma\, k^2 V_{kkk} = 0 \tag{4a}$$

$$V_{kk} = u''\, C_k \tag{4b}$$

$$V_{kkk} = u'''\, C_k\, C_k + u''\, C_{kk} \tag{4c}$$

Insert (3) and (4b) in (4a) to get

$$0 = u'\left(f_k - \rho\right) + u''\, C_k\, \left(f - C\right) + 2\,\sigma\, k\, u''\, C_k + \sigma\, k^2\left(u'''\, C_k\, C_k + u''\, C_{kk}\right) \tag{5}$$

Note the difference with Equation (6) in the Judd & Guu paper: this must be a typo in the paper. The Mathematica code uses a slightly different specification:

$$B\left(k,\sigma\right) \doteq \frac{u'}{u''}\left(f_k-\rho\right)+C_k\ \left(f-C+2\,\sigma\,k\right)+\sigma\,k^2\left(\frac{u'''}{u''}\,C_k\,C_k+\,C_{kk}\right)=0 \tag{6}$$

Auxiliary functions are specified for u'/u'' and u'''/u''. The steady state condition is

$$C\left(\bar{k},0\right)=f\left(\bar{k}\right) \tag{7}$$

where $\bar{k}$ indicates the capital stock k in the steady state (kss in the Judd & Guu paper). Optimal consumption will be a function of the actual stock of capital, k, and the variance, σ: $C=C\left(k,\sigma\right)$. We don't know the actual functional form but we approximate it with a Taylor polynomial:

$$\begin{aligned}\hat{C}\left(k,\sigma\right)=\bar{C}+\bar{C}_k\left(k-\bar{k}\right)+\bar{C}_\sigma\sigma+&\\ \frac{1}{2!}\left[\bar{C}_{kk}\left(k-\bar{k}\right)^2+\bar{C}_{\sigma\sigma}\sigma^2+2\bar{C}_{\sigma\,k}\sigma\left(k-\bar{k}\right)\right]+&\\ +\frac{1}{3!}\left[\bar{C}_{kkk}\left(k-\bar{k}\right)^3+\bar{C}_{\sigma\sigma\sigma}\sigma^3+3\bar{C}_{\sigma\sigma\,k}\sigma^2\ \left(k-\bar{k}\right)+3\bar{C}_{\sigma\,kk}\sigma^2\ \left(k-\bar{k}\right)\right]+\cdots=&\\ \sum_{i=0}^{n}\frac{1}{n!}\sum_{j=0}^{i}\binom{i}{j}\frac{\partial^i\bar{C}}{\partial\,k^j\partial\,\sigma^{i-j}}\left(k-\bar{k}\right)^j\ \sigma^{i-j}=\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{1}{j!(i-j)!}\frac{\partial^i\bar{C}}{\partial\,k^j\partial\,\sigma^{i-j}}\left(k-\bar{k}\right)^j\ \sigma^{i-j}&\end{aligned} \tag{8}$$

where $\bar{C}\doteq C\left(kss,0\right)$ indicates the consumption function C at the steady state. The coefficients to be estimated can be expressed in the following matrix:

$$\mathbf{C}=\begin{bmatrix}\bar{C} & \bar{C}_k & \bar{C}_{kk} & \ldots & \bar{C}_{k^{(n-2)}} & \bar{C}_{k^{(n-1)}} & \bar{C}_{k^{(n)}}\\ \bar{C}_\sigma & \bar{C}_{k\,\sigma} & \bar{C}_{kk\,\sigma} & \ldots & \bar{C}_{k^{(n-2)}\,\sigma} & \bar{C}_{k^{(n-1)}\,\sigma} & \\ \bar{C}_{\sigma\,\sigma} & \bar{C}_{k\,\sigma\,\sigma} & \bar{C}_{kk\,\sigma\,\sigma} & \ldots & \bar{C}_{k^{(n-2)}\,\sigma\,\sigma} & & \\ \ldots & \ldots & \ldots & \ldots & & & \\ \bar{C}_{\sigma^{(n-2)}} & \bar{C}_{k\,\sigma^{(n-2)}} & \bar{C}_{kk\,\sigma^{(n-2)}} & & & & \\ \bar{C}_{\sigma^{(n-1)}} & \bar{C}_{k\,\sigma^{(n-1)}} & & & & & \\ \bar{C}_{\sigma^{(n)}} & & & & & & \end{bmatrix} \tag{9}$$

The strategy to solve this is as follows:

1. Find $\bar{C}$ by the steady state condition and insert its value in $B\left(k,\sigma\right)$;
2. Find $\bar{C}_k$, ..., $\bar{C}_{k^{(n)}}$:
   (a) Find $\bar{C}_k$ through $\bar{B}_k$ (quadratic function; take positive solution) and insert the value in $B\left(k,\sigma\right)$;
   (b) Find $\bar{C}_{kk}$ through $\bar{B}_{kk}$ with the estimates of $\bar{C}$ and $\bar{C}_k$, and insert the value in $B\left(k,\sigma\right)$;

(c) And so on.

3. Find $\bar{C}_\sigma$ through $\bar{B}_\sigma$ with estimates of $\bar{C}_k$ and $\bar{C}_{kk}$;
4. Find estimates of $\bar{C}_{\sigma\,k}$, $\bar{C}_{\sigma\,kk}$, …:
   (a) Find $\bar{C}_{\sigma\,k}$ through $\bar{B}_{\sigma\,k}$ and insert the value in $B\,(k,\sigma)$;
   (b) Find $\bar{C}_{\sigma\,kk}$ through $\bar{B}_{\sigma\,kk}$ and insert the value in $B\,(k,\sigma)$;
   (c) And so on.
5. Find $\bar{C}_{\sigma\,\sigma}$ through $\bar{B}_{\sigma\,\sigma}$;
6. Find estimates of $\bar{C}_{\sigma\,\sigma\,k}$, $\bar{C}_{\sigma\,\sigma\,kk}$, …:
   (a) Find $\bar{C}_{\sigma\,\sigma\,k}$ through $\bar{B}_{\sigma\,\sigma\,k}$ and insert the value in $B\,(k,\sigma)$;
   (b) Find $\bar{C}_{\sigma\,\sigma\,kk}$ through $\bar{B}_{\sigma\,\sigma\,kk}$ and insert the value in $B\,(k,\sigma)$;
   (c) And so on.
7. Like steps 5 and 6 for the remaining terms associated with σ.

The residual function is estimated by inserting $\hat{C}$ in $B\,(k,\sigma)$ and dividing the expression by $\rho\,\frac{u'(\bar{C})}{u''(\bar{C})}$:

$$R \doteq \frac{\frac{u'}{u''}\,(f_k-\rho)+\hat{C}_k\,\left(f-\hat{C}+2\,\sigma\,k\right)+\sigma\,k^2\,\left(\frac{u'''}{u''}\,\hat{C}_k\,\hat{C}_k+\hat{C}_{kk}\right)}{\rho\,\frac{u'(\bar{C})}{u''(\bar{C})}} \tag{10}$$